B.Sc. Part III 5th Paper

Infinite Products (contd.)

Formula: * $\prod (1 \pm U_n)$ is convergent/divergent according as the series $\sum U_n$ is convergent/divergent.

Q. Prove that $\prod \left[1 + \left(\frac{nx}{n+1}\right)^n\right]$ is absolutely convergent if $|x| < 1$.

Soln. Let $\prod (1+U_n) = \prod \left[1 + \left(\frac{nx}{n+1}\right)^n\right]$

$$\Rightarrow U_n = \left(\frac{nx}{n+1}\right)^n =$$

$\because \prod (1+U_n)$ and $\sum U_n$ behave alike.

Applying Cauchy's root test, we have

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{nx}{n+1}\right)^n\right]^{1/n} = \lim_{n\to\infty} \frac{nx}{n+1}$$

$$= x \times \lim_{n\to\infty} \frac{n}{n+1} = x \times \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)$$

$$= x \times \frac{1}{1+0} = x$$

~~Hence~~ Here, $|x| < 1$

$\Rightarrow \sum |U_n|$ is convergent

$\Rightarrow \prod (1+U_n)$ is absolutely convergent.

Q. Test the convergence of the infinite product $\prod \left[\left(1+\frac{z}{n\pi}\right) e^{-z/n\pi}\right]$

Soln Let $\prod (1+U_n) = \prod \left(1+\frac{z}{n\pi}\right) e^{-z/n\pi}$

$\Rightarrow 1+U_n = \left(1+\frac{z}{n\pi}\right)\left(1 - \frac{z}{n\pi} + \frac{z^2}{\lfloor 2\, n^2\pi^2} - \frac{z^3}{\lfloor 3\, n^3\pi^3} + \cdots\right)$

$= 1 - \cancel{\frac{z}{n\pi}} + \frac{z^2}{\lfloor 2\, n^2\pi^2} - \frac{z^3}{\lfloor 3\, n^3\pi^3} + \cdots$

$+ \cancel{\frac{z}{n\pi}} - \frac{z^2}{n^2\pi^2} + \frac{z^3}{\lfloor 2\, n^3\pi^3} - \frac{z^4}{\lfloor 3\, n^3\pi^3}$

$\Rightarrow 1+U_n = 1 - \frac{z^2}{2n^2\pi^2} + \frac{z^3}{3n^3\pi^3} \cdots$

$\Rightarrow U_n = -\frac{1}{2}\frac{z^2}{n^2\pi^2} + \frac{z^3}{3n^3\pi^3} \cdots$

Let $\sum V_n = \sum \frac{1}{n^2}$ i.e. $\frac{|U_n|}{V_n} = \frac{-1}{2}\frac{z^2}{\pi^2} + \frac{z^3}{3n\pi^3} \cdots$

$\Rightarrow \underset{n\to\infty}{Lt} \frac{|U_n|}{V_n} = +\frac{1}{2}\frac{|z|^2}{\pi^2}$ = finite $\Rightarrow \sum U_n$ and $\sum V_n$ behave alike

but $\sum V_n$ is cgt $\Rightarrow \sum |U_n|$ is cgt

$\Rightarrow \prod 1+U_n$ is abs. cgt.